Introduction to Special Relativity

Based upon Bernard Schutz's *A First Course in General Relativity*: Second Edition

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Contents

Chapter 1

Special Relativity

1.1 Basic Principles of Special Relativity

Albert Einstein proposed the theory of special relativity (SR) in 1905. The theory of special relativity was proposed in the paper "*On the Electrodynamics of Moving Bodies.*" The inconsistency of Maxwell's equations of electromanetism and Newtonian mechanics as well as the lack of a luminiferous aether led to the proposal of special relativity. Special relativity described the relationship between space and time and objects traveling at velocities close to the speed of light in the absence of gravitational fields.

Special Relativity is based upon two fundamental postulates:

1 Principle of Relativity: It is not possible to find the absolute velocity of an observer as his velocity depends on the use of a reference frame. Thus, the laws of physics stay the same in all inertial frames of reference. In other words, one can not calculate their velocity without using something else to act as a reference point.

2 Speed of Light: The speed of light c is approximately 3 x 10^8 m/s for all observers. Regardless of the state of motion of the observers relative to a light source or each other, the observers will measure the speed of the same photons to always be c

1.2 Inertial Observer in Special Relativity

The term "observers" is very common in relativity. It refers to a data collector. Specifically, an *observer* refers to a information-gathering system rather than an human individual. An *inertial observer* records information for a location (x, y, z) at a time (t). An observer must satisfy the three following conditions to be considered an interial observe:

1 The distance between two points is independent of time

2 Clocks, any device to measure the passage of time, that are located at regular increments are synchronized and run at the same rate

3 Space has a Euclidean geometry at any constant time. Euclidean geometry is the type of geomtry that you see in day to day life.

An *observation* by an inertial observer is defined as recording the location of an event (x, y, z) and recording the time on the clock at x, y, z. An inertial observer is also known as an *inertial reference frame.*

1.3 Natural Units

To help simplify measurements, c will now be defined as having a value of 1 rather than 3 x 10^8 m/s. Time will also be defined in meters rather than seconds. Similar to how a measure of time is a light-year (The distance light travels in one year), we can use *light-meters* as a measure of time. The distance light travels over a time interval of one meter is also one meter.

$$c = \frac{Distance\ Light\ Travels\ Over\ a\ Time\ Interval}{Time\ Interval}$$

$$c = \frac{1m}{1m} = 1$$

Since, light has traveled one meter over a time interval of one meter, we can simply the speed of light as "1". Since c is now defined as 1, we can use the following relationship

$$c = 3 \times 10^8\ m/s = 1$$

$$1\ s = 3 \times 10^8 m$$

$$1\ m = \frac{1}{3 \times 10^8} s$$

1.4 Spacetime Diagrams

Spacetime Diagrams are a tool to describe the relationships between position and time. Since it is difficult to visualize and draw a 4D diagram, it is easier to set one or two spatial values (x, y, or z) to zero and then draw the remaing dimensions. Figure 1.1 shows a slice of spacetime, the t-x plane. The spacial dimensions of y and z have been set to zero. Also, the following definitions will be essential to our study of special relativity:

1 Event: A single point in space at a fixed position in space and a fixed t in which something happens. This can refer to anything such as a detector being activated, a light bulb turning on, an object moving from rest, etc.

2 World Line: A world line of a particle represents its position in space at different times. It's essentially the path that an object takes in space through the passage of time.

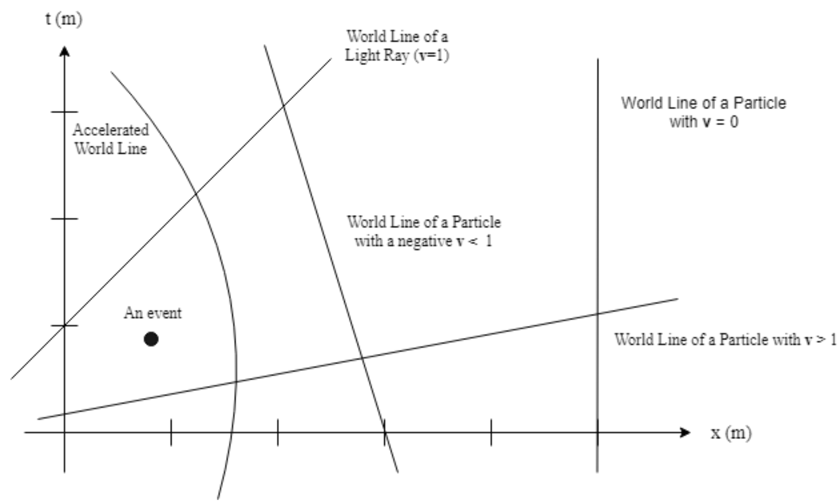

Figure 1.1: Spacetime Diagram of World Lines for Different Velocities

This diagram depicts the world lines of objects at various velocities. Note the units for the spacetime time graph. Both position and time are in meters since both time and position can be meters through the use of natural units. Unlike usual kimematic graphs, time is on the y axis rather than the x axis as well in a time-position graph. Under natural units, the speed of light is also defined as 1, therefore the velocities depicted in the spacetime graph does not refer to 1 m/s. Since the speed of light is defined as 1, a light ray has a world line of 45° as depicted by the diagram.

Similar to how the slope of a position-time graph gives velocity, we can apply the same concepts to a time-position graph.

Slope for a Position-Time Graph

$$v = \frac{x}{t} \tag{1.1}$$

$$v = \frac{x_1 - x_2}{t_1 - t_2} = \frac{dx}{dt} \tag{1.2}$$

This equation is very frequent when learning kinematic formulas. If you notice, this equation applies for a position-time graph, however a spacetime graph is defined as a time-position graph in which time is the y axis and x position is the x axis. Since the axis flip, the equation must flip as well. Therefore the slope for a spacetime graph does not give v, but rather:

$$\frac{1}{v} = \frac{dx}{dt} \tag{1.3}$$

This is the slope of a particle's world line in a space-time graph and its relationship to velocity.

1.5 Conventions in Special Relativity

In special relativity, the following conventions are frequently used

1 Events are denoted by cursive capital letters such as $\mathscr{A}$,$\mathscr{B}$,$\mathscr{C}$. The capital letter $\mathscr{O}$ does not indicate events, but rather it indicated observers.

2 Coordinates are defined as the following (t,x,y,z). Time is defined by t whereas space is defined by x,y,z. For example, (1, 3.2, 7, -2.7) refers to t=1, x=3.2, y=7, z=-2.7. Remember that since we use natural units, time is measured in meters. The spatial components x,y,z are measured in meters as well.

3 The coordinates (t,x,y,z) can also be refered to as (x^0,x^1,x^2,x^3). The superscripts are not exponents, but rather they act as labels. Similar to how subscripts were used to keep track of points (e.g. x_1 or x_A), superscripts are used the same way. These labels are kown as indices. Since these superscripts act as labels rather than exponents, $(x^2)^3$ refers to the y^3 and not x^6.

4 Latin and Greek indices are frequently use to assist in labeling coordinates. A common conventition is that the Greek alphabet such as μ and υ is used to describe both space and time components. Greek indices take values of 0, 1, 2, or 3. On the other hand, the Latin alphabet such as i and j is reserved for spatial components only. Latin indices take values of 1, 2 or 3.

1.6 Coordinate System Relative to Another Observer

Since all observers detect the same events (in the same spacetime), we can draw coordinate systems relative to each observer. For example, an observer $\mathscr{O}$ has the coordinates *(t,x)*. Another observer $\overline{\mathscr{O}}$ is moving with a velocity v in the x direction. Thus his coordinates are ($\bar{t}$,$\bar{x}$). If we ignore the spatial components y and z by setting them to zero, the time-position plane for $\mathscr{O}$ is the following:

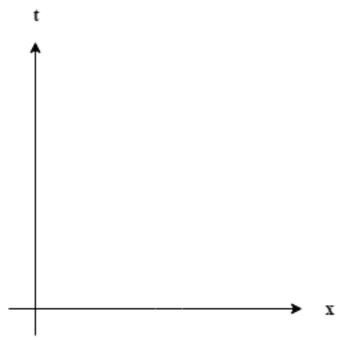

Figure 1.2: Time-Position Diagram For Observer $\mathscr{O}$

From $\mathscr{O}$'s point of view, $\overline{\mathscr{O}}$ appears to be moving. From $\overline{\mathscr{O}}$'s point of view, $\mathscr{O}$ appears to be moving. Therefore for $\mathscr{O}$, as time passes (t), his position does not change, but he notices that $\overline{\mathscr{O}}$'s position is changing. $\overline{\mathscr{O}}$ belives that as time passes for him ($\bar{t}$), his position does not change ($\bar{x}$). According to kinematics, x = v × t. Therefore, from $\mathscr{O}$'s perspective, $\overline{\mathscr{O}}$ has a velocity. If this is difficult to grasp, pretend an individual named Isabella is on a plane while her friend Emma is on the ground. From Emma's perspective, the plane with Isabella is moving in the sky. When Isabella looks out the window, she sees that the ground beneath her (Emma) is moving. In this case, Emma is $\mathscr{O}$ whereas Isabella is $\overline{\mathscr{O}}$. Velocity is relative!

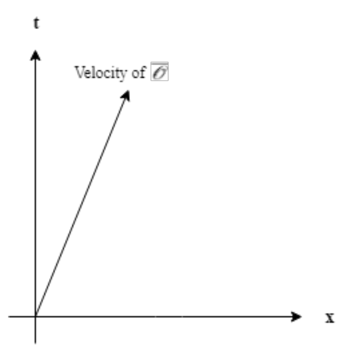

Figure 1.3: Velocity of $\overline{\mathscr{O}}$ relative to $\mathscr{O}$

From $\overline{\mathscr{O}}$'s perspective, as time passes ($\bar{t}$), his position stays the same ($\bar{x}$). Therefore as $\bar{t}$ passes, $\bar{x}$ is equal to 0 for $\overline{\mathscr{O}}$'s perspective. Thus the velocity line that $\mathscr{O}$ saw, acts as the $\bar{t}$ axis since position ($\bar{x}$) stays zero as $\bar{t}$ passes for the observer $\overline{\mathscr{O}}$.

Refer back to the airplane example. From Emma's perspective, her friend on the plane is moving with a velocity. From Isabella's perspective, Emma is moving on the ground while her plane is still in the air. Emma's velocity line that she observes is the same as Isabella's time axis because as time passes for Isabella, her position ($\overline{x}$) stays zero.

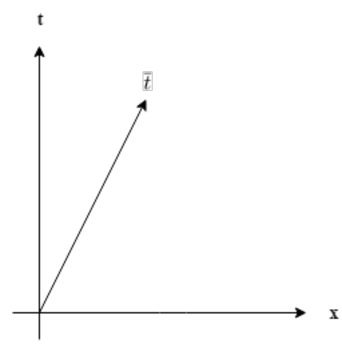

Figure 1.4: $\bar{t}$ axis

We have found the location of the $\bar{t}$ axis, now we need to find the $\overline{x}$ axis.

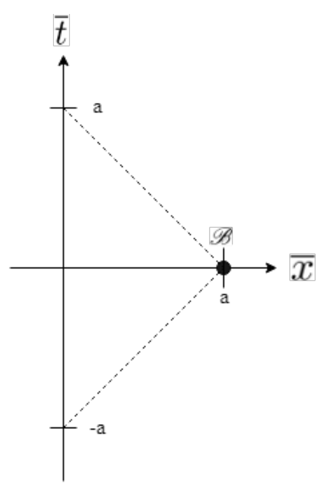

Figure 1.5: Light Ray for Observer $\overline{\mathscr{O}}$

Since c has a value of 1 as defined by natural units, the slope of the light ray is one, thus the amount of time that passes is equal to the distance traveled by the light ray. On Figure 1.5, a light ray originated at a position zero (for observer $\overline{\mathscr{O}}$) at a time of -a. Over the time interval of -a to 0, the light ray travels a distance of a. At event $\mathscr{B}$, the ligh ray is reflected back towards its origin. From the time interval 0 to a, the light ray returns back to its origin where $\overline{x}$ equals 0. You can clearly see in the diagram that light travels at 45°.

Now lets say that two light rays are released. One is released at "a" whereas the other one is released at "-a." If you track the path of the light rays, both of them would intersect at Event $\mathscr{B}$ as depicted by the diagram. This indicates that for $\overline{\mathscr{O}}$, the $\overline{x}$ and $\bar{t}$ axis are perpendicular to each other.

The following is the spacetime diagram for $\mathscr{O}$.

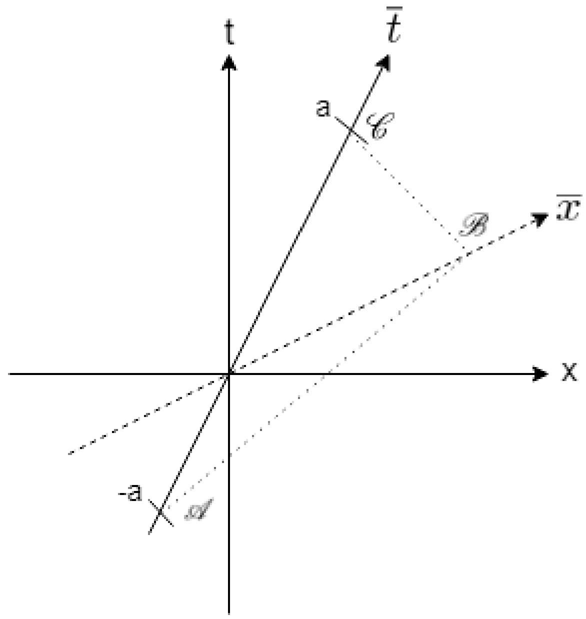

Figure 1.6: Light Ray for Observer $\overline{\mathscr{O}}$

First, lets explain how the $\overline{x}$ axis was created. If you recall from Figure 1.5, the light rays were traveling at 45°. Therefore, when you draw two light rays originating from -a and a, you end up with these light rays intersecting at $\mathscr{B}$. Then we draw a line from the origin through $\mathscr{B}$ which gives us our $\overline{x}$. Now what does $\overline{x}$ axis represent? The $\overline{x}$ axis represents the various points that an object can be located at for $\bar{t} = 0$.

Since the $\overline{x}$ and $\bar{t}$ axis are perpendicular to each other, as seen in Figure 1.5, the angle between the $\bar{t}$ and t axis is the same as the angle between $\overline{x}$ and x axis. The following figure shows these angles[1].

[1]Figure obtained from: *A First Course in General Relativity: Second Edition* by Bernard Schutz.

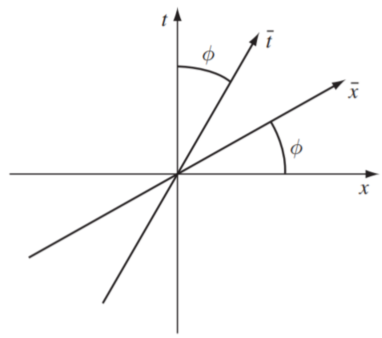

Figure 1.7: Spacetime Diagram of $\mathscr{O}$

From $\mathscr{O}$'s point of view, $\overline{\mathscr{O}}$ is moving to the right. Therefore $\overline{\mathscr{O}}$ sees that $\mathscr{O}$ is moving to the left. The spacetime diagram for $\overline{\mathscr{O}}$ is similar to $\mathscr{O}$ except it is towards the left. Figure 1.8[2] depicts this diagram.

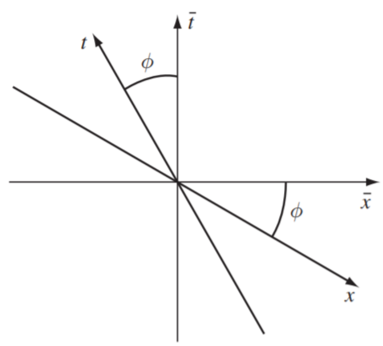

Figure 1.8: Spacetime Diagram of $\overline{\mathscr{O}}$

These diagrams will be used later on to assist in deriving the Lorentz Contraction.

1.7 The Spacetime Interval

As we know, at high velocities, there is the bending of spacetime. Therefore we can no longer use the traditional way to measure distance in 3D space through the Pythagorean theorem which is

$$(\Delta d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

[2]Figure obtained from: *A First Course in General Relativity: Second Edition* by Bernard Schutz.

Now we need to account for the bending of spacetime. Since light is measured at a constant velocity of c regardless of observers, it can be used to help create a formula to measure the spacetime interval ($(\Delta s)^2$), the seperation between two events by time and space.

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

The value of the spacetime interval is the same for all observers regardless of their motion. Any observer may calculate a different value of Δt[3], Δx, Δy, or Δz, but the spacetime interval will always stay the same for the an event.

Usually, we refer to the spacetime interval as $(\Delta s)^2$ rather than as (Δs). The spacetime interval can be positive, zero, or negative:

1 A positive spacetime interval is known as being spacelike. In a spacelike interval, an object can only be present at two certain events (locations) if it travels at a velocity greater than the speed of light.

2 A negative spacetime interval is known as being timelike. In a timelike interval, an object can be present at two events if it travels at a speed less than the speed of light.

3 If the spacetime interval equals zero, then the interval is called lightlike. Finally, during a lightlike event, an object can only be present at two events if it travels at the speed of light. If two events are too far apart, an object can not reach the event unless it has more time to reach it.

We can also use the spacetime interval equation to reinforce the concept of the speed of light. A lightlike interval is when Δs^2 is zero, so we have the following equation:

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$0 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta t)^2$$

$$\frac{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}{(\Delta t)^2} = 1$$

From the Pythagorean Theorem, we know $(\Delta d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

[3] Recall that since we are using natural units, $(\Delta t)^2$ is measured in meters. If we were not using natural units, the term $(\Delta t)^2$ would be $(\Delta ct)^2$. Both $(\Delta t)^2$ and $(\Delta ct)^2$ describe the distance light travels over a specific time interval.

$$v^2 = \frac{(\Delta d)^2}{(\Delta t)^2}$$

$$v = \pm\frac{\Delta d}{\Delta t}$$

If we use 1m for distance and 1m for time (natural units), we get:

$$v = \frac{1m}{1m} = \pm 1$$

We have demonstrated that the speed of light is indeed 1 according to the lightlike spacetime interval.

The spacetime interval is an important concept in understanding special relativity. All observers, regardless of their motion, will measure the spacetime interval to be the equal to each other for the same two events ($\Delta\bar{s}^2 = \Delta s^2$). Although physical distance and the passage of time may change due to relativistic speeds, the spacetime interval will always stay the same for an event.

1.7.1 Spacetime Interval and Geometry

Lets prove that $\Delta\bar{s}^2 = \Delta s^2$ or in other words, prove that the spacetime interval stays the same for two observers for an event. This is also known as the invariance of the spacetime interval. Initially, let's use geometry to help visualize the invariance of the spacetime interval.

If you recall, the Pythagorean Theorem is $a^2 + b^2 = c^2$ or rather $\Delta a^2 + \Delta b^2 = \Delta c^2$.

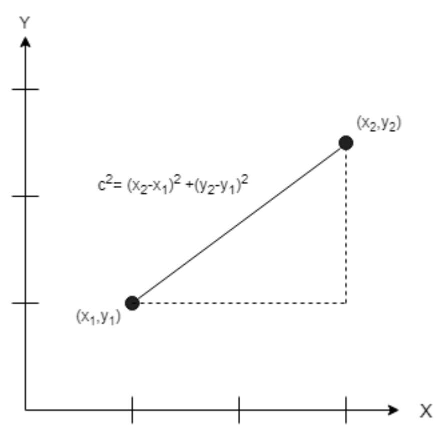

Figure 1.9: The Pythagorean Triangle

Now lets treat this as a vector from the origin rather than a line between two points.

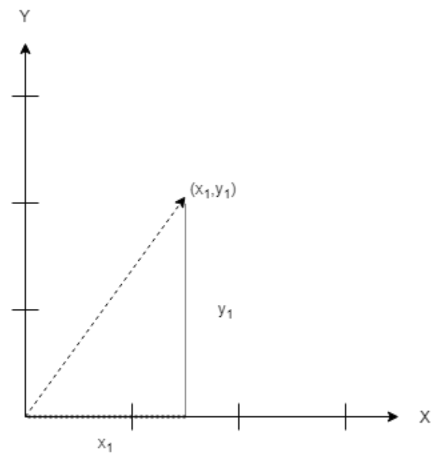

Figure 1.10: A Vector Starting at the Origin

As you can see, the vector can be split into an x and y component. If you use the Pythagorean Theorem, you end up with the magnitude of the vector. We can simulate the presence of another observer by rotating, translating, etc. the graph. For this problem, lets rotate the axis to indicate another observer.

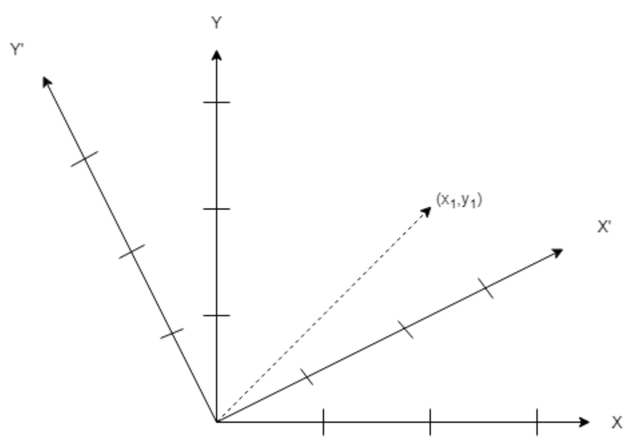

Figure 1.11: Rotation of Axis to Simulate Another Observer's Frame of Reference

Due to the rotation, both observers have a different frame of reference. Each observer calculates a different value of x and y.

Although they have a different frame of reference, the magnitude of the vector stays the same for both observers. Note that the angle between the y and y' axis as well as the x and x' axis is the same. The x' and y' axis are perpendicular to each other which is why the angles are the same.

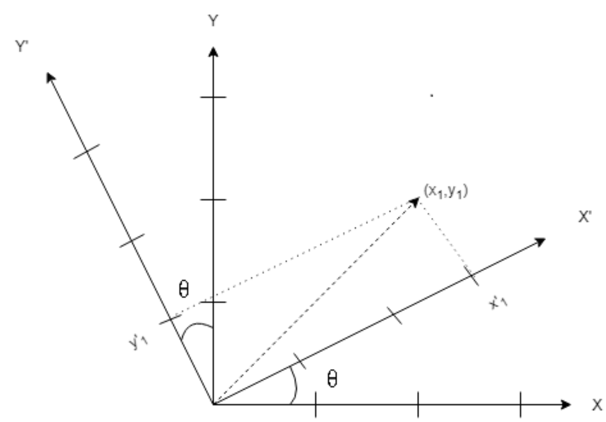

Figure 1.12: The Second Observer Calculates Different Values for x and y

The angle between the vector and the x' axis is some unknown number known as a. Now we use trigonometry.

$$\sqrt{{x_1}^2 + {y_1}^2} cos(a) = x'_1$$

$$\sqrt{{x_1}^2 + {y_1}^2} sin(a) = y'_1$$

We know that the magnitude of the vector equals $\sqrt{{x_1}^2 + {y_1}^2}$. The formula for the magnitude of the vector for the second observer is the same as well $\sqrt{{x'_1}^2 + {y'_1}^2}$.

Now, we plug in the values for x'_1 and y'_1 that we obtained from trigonometry.

$$\sqrt{{x'_1}^2 + {y'_1}^2}$$

$$\sqrt{(\sqrt{{x_1}^2 + {y_1}^2}\ cos(a))^2 + (\sqrt{{x_1}^2 + {y_1}^2}\ sin(a))^2}$$

$$\sqrt{({x_1}^2 + {y_1}^2)\ cos^2(a) + ({x_1}^2 + {y_1}^2)\ sin^2(a)}$$

$$\sqrt{({x_1}^2 + {y_1}^2)(cos^2(a) + sin^2(a))}$$

Trig Identity: $sin(a)^2 + cos(a)^2 = 1$

$$\sqrt{({x_1}^2 + {y_1}^2) \times 1}$$

The magnitude of the vector is the same for both observers.

The magnitude of a vector stays the same for not only 2D space, but 3D space as well. Since it is difficult to visualize and draw 4D space, you can set one of the spatial dimensions (x, y or z) to zero and have time as one of the three axis. As a result, you essentially have a three dimensional figure as seen below.

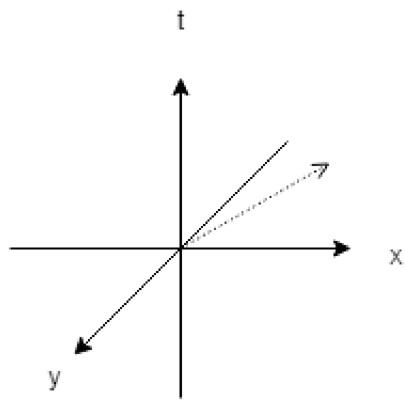

Figure 1.13: X, Y and Time

Similar to how a change of the reference frame does not affect the magnitude of a vector, this same principle can be applied to the four dimensional spacetime.

By changing the frame of reference, each observer can measure different values for x, y, z, and t, however the magnitude (Δs^2) will remain the same.

1.7.2 Derivation of Invariance of the Interval

Recall that the spacetime interval is:

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Since the speed of light is 1 in natural units, the spacetime interval equals zero. As a result, the differences in (Δt, Δx, Δy, and Δz) for $\mathscr{A}$ and $\mathscr{B}$ is the relation $-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 0$. Since all observers measure the speed of light to be the same, observer $\overline{\mathscr{O}}$ satisfies the relation $-(\Delta \bar{t})^2 + (\Delta \overline{x})^2 + (\Delta \overline{y})^2) + (\Delta \overline{z})^2 = 0$. Assume that the origin for the coordinates of $\mathscr{O}$ and $\overline{\mathscr{O}}$ are the same ($\bar{t} = \overline{x} = \overline{y} = \overline{z} = 0$) and ($t = x = y = z = 0$).

We can write

$$\Delta \bar{s}^2 = -(\Delta \bar{t})^2 + (\Delta \overline{x})^2 + (\Delta \overline{y})^2 + (\Delta \overline{z})^2$$

As a tensor,

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \mathbf{M}_{\alpha\beta} (\Delta x^\alpha)(\Delta x^\beta)^4$$

Suppose $\Delta t = \Delta r$ and $\Delta s^2 = 0$. We get:

$$0 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2) + (\Delta z)^2$$

$$(\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2) + (\Delta z)^2$$

$$\Delta r = \Delta t = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

The matrix for the tensor is the following

$$\begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Using the symmetry of the matrix and the relationship for Δr. We get:

$$\Delta \bar{s}^2 = \mathbf{M}_{00}(\Delta r)^2 + 2(\sum_{i=1}^{3} \mathbf{M}_{0i} \Delta x^i)\Delta r + \sum_{i=1}^{3} \sum_{j=1}^{3} \mathbf{M}_{ij} \Delta x^i \Delta x^j$$

We can show that $\mathbf{M}_{0i} = 0$ for i = 1,2,3 by emitting a beam of light along the x axis that starts at the origin.

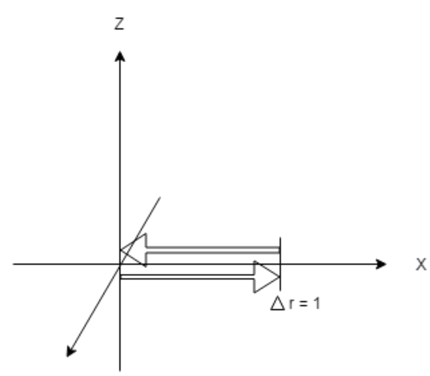

Figure 1.14: Beams of Light Along the X Axis

[4]Remember that $\Delta x^0 = t$, $\Delta x^1 = x$, $\Delta x^2 = y$, and $\Delta x^3 = z$

This beam travels for $\Delta r = \Delta t = 1$. Since it is a light ray, the spacetime interval equals 0 as well. The beam of light that travels from the origin to $\Delta r = 1$ has the equation:

$$0 = \mathbf{M}_{00} + 2\mathbf{M}_{01} + \mathbf{M}_{11}$$

For the beam that travels from $\Delta r = 1$ to the origin:

$$0 = \mathbf{M}_{00} - 2\mathbf{M}_{01} + \mathbf{M}_{11}$$

Now, lets add the the values for M_{01}.

$$0 = \mathbf{M}_{00} + 2\mathbf{M}_{01} + \mathbf{M}_{11} \rightarrow 0 = -\mathbf{M}_{00} + 2\mathbf{M}_{01} - \mathbf{M}_{11}$$

$$0 = \mathbf{M}_{00} - 2\mathbf{M}_{01} + \mathbf{M}_{11}$$

$$0 = 4\mathbf{M}_{01}$$

$$\mathbf{M}_{01} = 0$$

The emission of the right ray can be extended to both the y and z axis therefore $\mathbf{M}_{0i} = 0$ for i = 1,2,3.

With the Kroneck delta (δ_{ij}) defined as:

$$\begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

We can write

$$\mathbf{M}_{ij} = -(\mathbf{M}_{00})\delta_{ij} \quad (i, j = 1, 2, 3)$$

Therefore,

$$\Delta \bar{s}^2 = \mathbf{M}_{00}[(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2]$$

.

Lets define a function:

$$\phi(\mathbf{v}) = -\mathbf{M}_{00}$$

As a result,

$$\Delta \bar{s}^2 = \phi(\mathbf{v})\Delta s^2$$

Now we need to prove that $\phi(\mathbf{v})$ is equal to 1. There are two parts of the proof to show that $\phi(\mathbf{v})$ is equal to 1. For the first part of the proof, we need to show that $\phi(\mathbf{v}) = \phi(|\mathbf{v}|)$.

There are two observers, $\mathscr{O}$ and $\overline{\mathscr{O}}$. $\overline{\mathscr{O}}$ is moving at a velocity $\mathbf{v}$ in the x direction relative to $\mathscr{O}$. Figure 1.15[5] shows a rod that is along the y axis (Perpendicular along the y axis). The diagram does not depict the z axis since

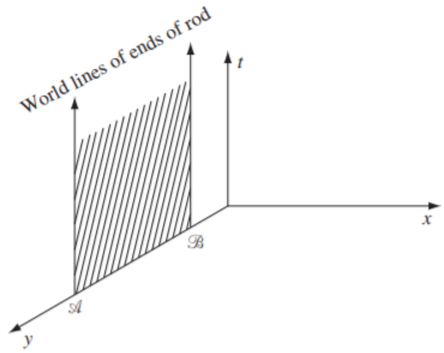

Figure 1.15: Rod Along The Y Axis

it equals zero. The world lines at the end of the rod are shown as well. You can see at t=0, x = 0 as well.

Therefore the square of the lenght of the rod is the same as the interval between the two events $\mathscr{A}$ and $\mathscr{B}$.

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

At t=0, both x and z are equal to 0.

Thus

$$(\Delta s)^2 = (\Delta y)^2$$

At t=0, both events $\mathscr{A}$ and $\mathscr{B}$ are simultaneous. These events are simultanteous when measured by $\overline{\mathscr{O}}$ as well. Figure 1.16[6], shows a clock in $\overline{\mathscr{O}}$. The world line of the clock is perpendicular to the y axis and parallel to the t-x plane and $\bar{t}$ axis. Refer to figure 1.7 to get a better idea of the orientation relative to the $\bar{t}$ axis. This clock that we have selected emits light rays at event $\mathscr{P}$. The light rays travel to $\mathscr{A}$ and $\mathscr{B}$. If you look at the geometric orientation of figure 1.16, you can see that the light from the two events arrives at $\mathscr{L}$ at the same time.

As a result, the two events appear simultaneous to $\overline{\mathscr{O}}$. If $\mathscr{A}$ and $\mathscr{B}$ are simultaneous in $\overline{\mathscr{O}}$ then the interval between the two events is also the square of the lenght of the rod in $\overline{\mathscr{O}}$.

$$\Delta\overline{L}^2 = \phi(\mathbf{v})\Delta L^2$$

Since the rod is perpendicular to the velocity of $\overline{\mathscr{O}}$, the direction of motion does not matter. The reference frames can also be oriented in any direction in spacetime since it is not possible to indicate one's orientation without using a point of reference. Therefore,

[5]Figure obtained from: *A First Course in General Relativity: Second Edition* by Bernard Schutz

[6]Figure obtained from: *A First Course in General Relativity: Second Edition* by Bernard Schutz

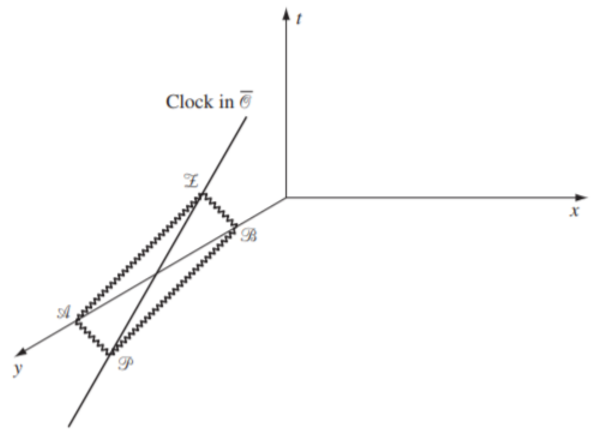

Figure 1.16: Clock in $\overline{\mathscr{O}}$

$$\phi(\mathbf{v}) = \phi(|\mathbf{v}|)$$

For the second part of the proof, we will use the equation $\Delta\bar{s}^2 = \phi(\mathbf{v})\Delta s^2$. There are three frames: $\mathscr{O}, \overline{\mathscr{O}}$ and $\overline{\overline{\mathscr{O}}}$. $\overline{\mathscr{O}}$ moves at a velocity $\mathbf{v}$ relative to $\mathscr{O}$. $\overline{\overline{\mathscr{O}}}$ moves at -$\mathbf{v}$ relative to $\overline{\mathscr{O}}$. You can see that $\mathscr{O}$ and $\overline{\overline{\mathscr{O}}}$ are essentially the same.

Therefore:

$$\Delta\bar{\bar{s}}^2 = \phi(\mathbf{v})\Delta\bar{s}^2$$
$$\Delta\bar{s}^2 = \phi(\mathbf{v})\Delta s^2$$

Using substitution,

$$\Delta\bar{\bar{s}}^2 = \phi(\mathbf{v})\phi(\mathbf{v})\Delta s^2$$
$$\Delta\bar{\bar{s}}^2 = (\phi(\mathbf{v}))^2\Delta s^2$$

Since $\mathscr{O}$ and $\overline{\overline{\mathscr{O}}}$ are equal to each other, $\Delta\bar{\bar{s}}^2$ is equal to Δs^2 as well. Therefore,

$$\phi(\mathbf{v}) = \pm 1$$

Although $\phi(\mathbf{v})$ can equal plus or negative 1, we can get rid of the negative sign by the following scenario.

Imagine that $\overline{\mathscr{O}}$ is moving at a $+\mathbf{v}$ in the x direction relative to $\mathscr{O}$.

$$\Delta\bar{s}^2 = \phi(\mathbf{v})\Delta s^2$$

As the velocity becomes smaller and smaller, the two reference frames start to become the same and therefore $\phi(\mathbf{v})$ can not equal a negative value.

$$\lim_{v\to 0}\phi(\mathbf{v}) = 1$$

Consequently,

$$\Delta\bar{s}^2 = \Delta s^2$$

1.8 Minkowski Space

Minkowski Space is an essential topic of special relativity. Minkowski Space is the combination of the three dimensional Euclidean space and time in four-dimensions. Minkowski spacetime only applies to special relativity and not general relativity. General relativity describes relativity under the effects of a gravitational field which creates a curved spacetime fabric. Minkowski spacetime is also called flat spacetime as it refers to a spacetime fabric that is unstretched by the effects of gravity. Figure 1.17 and 1.18[7] show the difference between curved and flat spacetime.

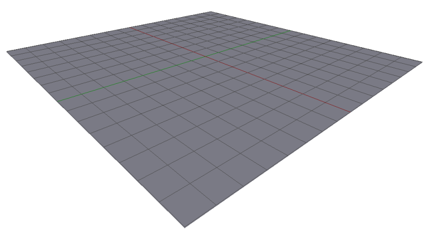

Figure 1.17: Flat Spacetime (Minkowski Spacetime)

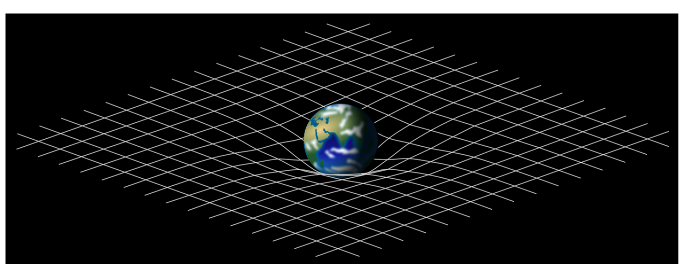

Figure 1.18: Curved Spacetime (Spacetime in General Relativity)

[7] Fetched from: https://en.wikipedia.org/wiki/General_relativity#/media/File:Spacetime_lattice_analogy.svg

1.8.1 Minkowski Metric

The Minkowski Metric is a very common tensor in special relativity. It is defined as

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The spacetime interval also equals:

$$ds^2 = (\Delta t \;\; \Delta x \;\; \Delta y \;\; \Delta z) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix}$$

Normally, when we calculate differences in physical distances, we find Δ x, Δ y, and Δ z. For spacetime, we can no longer resort to just the 3 dimensions. The Minkowski Tensor is used as a tool to find distortions in spacetime.

For example, imagine that there is a ruler in a gravitational wave detector. The goal of this ruler is to measure changes in physical distance in the light beams that are traveling down the arms of the gravitational wave detector. When a gravitational wave passes through the observatory, both the light beams *and* the ruler experiences a change in lenght. Since both the ruler and light deform, the ruler is not viable in measuring the contraction. Thus, we need a different tool. The Minkowski Metric is the tool that is used to measure such contractions.

1.8.2 Light Cones

Light Cones are a way to demonstrate the path light would take in spacetime if it travels in all directions and starts at a single point in spacetime. Look at the following figure:

Since light has a finite speed (3×10^8 m/s), there is a limit to how much distance can be covered in a specific time frame. Since light is also the fastest speed possible, the maximum speed limit of the universe is the speed of light. There is therefore a limitation on how fast information can travel: the speed of light. The red cone represents the maximum distance that information can be gathered or transmitted over a certain time interval.

Assume that you are an observer at time equals zero (Your reference point). One year has passed. You can only see events that are within one light year of your position because nothing can travel faster than light. If you want to observe a supernova that is two light years away, you need to wait two years before the light from the supernova can reach you. Your light cone needs to grow bigger via the passage of time. Similarly, assume that you transmit a signal to another planet that is 2 light years away. It will two years for the signal to reach the planet as nothing can travel faster than light.

As you can see in the diagram, as time passes, the size of the cone expands. Negative time merely refers to the past relative to a reference point (Ex. The

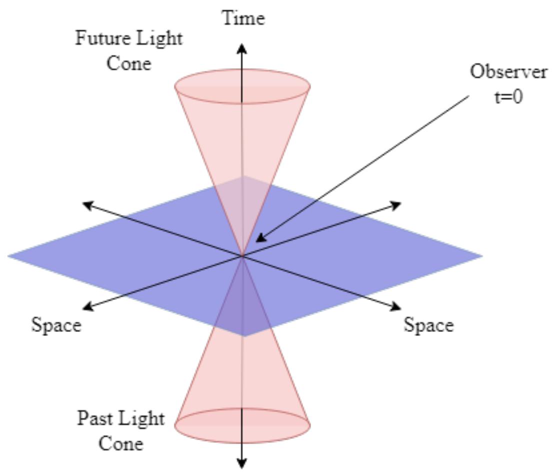

Figure 1.19: Lightcone Diagram Depicting the Future and Past Light Cone

year 2000 is in the past relative to the present). The area inside of the cone represent how far something can travel through spacetime if it travels at a speed less than c. The boundaries of the light cone represent the path that light would take through spacetime. Anywhere not in the light cone is unreachable unless more time passes and the light cone expands.

As a result, multiple events can have a common past and future as you can see in Figure 1.17[8].

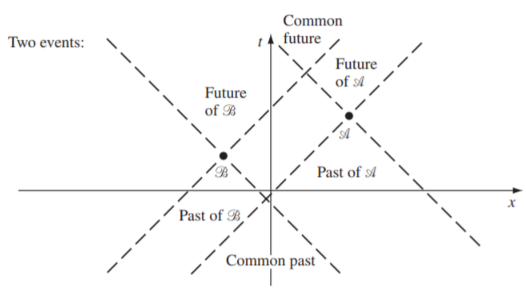

Figure 1.20: The Common Past and Future of Two Events

Assume that a flash of light is released at event $\mathscr{A}$ and $\mathscr{B}$. Both flashes of light are only detectable at a certain time if you were to travel to the common future that these two events share.

The speed of light also places a limit on how fast information can travel in the universe. As the universe expands faster than light, some pieces of infor-

[8]Figure obtained from: *A First Course in General Relativity: Second Edition* by Bernard Schutz

mation will become undetectable as the light from certain events will be unable to reach Earth. You may ask: How is this possible? Nothing could travel faster than light. This statement is true, however c is a limit on how fast things can move *through* the universe, but it does not place a limit on how fast the universe can expand itself. A speed limit sign on the road limits how fast cars can move on the road. It does not describe how fast the road itself can stretch

The spacetime interval can be extended to the concept of light cones as well. If you recall Δs^2 can be positive, negative, or zero.

1 When Δs^2 is positive, it means that the serperation between the events extends outside of the lightcone. As a result, these spacelike seperated events are undetectable for that time frame

2 If Δs^2 is negative, then these events are inside of the light cone. Thus, these timelike seperated events can be detected at a velocity less than the speed of light.

3 If Δs^2 is zero, then the events are lightlike seperated. They can only be detected by moving at the space speed as light. The event is on the boundary of the light cone.

Light cones are a great visual tool to demonstrate the propogation of light in spacetime as well as common times between events.

1.9 Time dilation and the Lorentz Contraction/Transformation

The Lorentz Contraction is defined to be equal to $\sqrt{1-v^2}$. Recall that this contraction is in natural units. The Lorentz Contraction expressed in SI units would be equal to $\sqrt{1-\frac{v^2}{c^2}}$. The Lorentz Contraction describes the passage of time and physical distance at relativistic velocities. For example, assume that a 1m long object with a clock inside of it is moving at 90% the speed of light in the x direction for 50 seconds. Under Newtonian physics, the clock would indicate that 50 seconds passed by and there would be no contraction of the 1m long object. Through the Lorentz Contraction, we get:

$$x = x_0 \times \sqrt{1-v^2}$$
$$t = t_0 \times \sqrt{1-v^2}$$

Now if we plug in the numbers from the problem, we get:

$$x = 1m \times \sqrt{1-.90^2}$$
$$t = 50s \times \sqrt{1-.90^2}$$

x = 0.436 m t = 21.794 s

The Lorentz Contraction plays an essential role when describe the conditions of an observer.

1.10 Lorentz Contraction Derivation

To begin deriving the Lorentz transformation, imagine the following. An observer $\overline{\mathscr{O}}$ moves with a speed v on the positive x axis relative to $\mathscr{O}$. Since movement is in the x direction, lenghts perpendicular to the x axis stay the same and do not contract.

Thus: y = $\overline{y}$ and z = $\overline{z}$. As moves, time and the x axis must change by some unknown amount.

$$\bar{t}\ axis(\overline{x}=0) : \alpha t + \beta x = 0$$

$$\overline{x}\ axis(\bar{t}=0) : \gamma t + \sigma x = 0$$

If you refer to Figures 1.7 and 1.8, the $\bar{t}$ and $\overline{x}$ axis have the following equations:

$$\bar{t}\ axis : vt - x = 0$$
$$\overline{x}\ axis : vx - t = 0$$

Using the above equations...

$$\begin{aligned} \bar{t}\ axis : vt - x &= 0 \\ vt &= x \\ v &= \frac{x}{t} \\ \alpha t + \beta x &= 0 \\ -\beta x &= \alpha \\ \frac{\beta}{\alpha} &= \frac{-t}{x} \\ \frac{\beta}{\alpha} &= -v \end{aligned}$$

$$\begin{aligned} \overline{x}\ axis : vx - t &= 0 \\ vx &= t \\ v &= \frac{t}{x} \\ \gamma t + \sigma x &= 0 \\ -\sigma x &= \gamma t \\ \frac{\sigma}{\gamma} &= \frac{-t}{x} \\ \frac{\sigma}{\gamma} &= \frac{-1}{v} \\ \frac{\gamma}{\sigma} &= -v \end{aligned}$$

Using substitution:

$$\begin{aligned} \bar{t}\ axis(\overline{x}=0) : \alpha t + \beta x &= 0 \\ \beta &= -vx \\ \bar{t} &= \alpha t - \alpha vx \\ \bar{t} &= \alpha(t - vx) \end{aligned}$$

$$\begin{aligned} \overline{x}\ axis(\bar{t}=0) : \gamma t + \sigma x &= 0 \\ \gamma &= -v\sigma \\ \overline{x} &= -\sigma vt + \sigma x \\ \overline{x} &= \sigma(x - vt) \end{aligned}$$

Figure 1.6 also shows that the events at ($\bar{t} = 0, \overline{x} = a$) and ($\bar{t} = a, \overline{x} = 0$) are also connected by the same light ray therefore the slopes of the two equations are the same and $\alpha = \sigma$.

We end up with the following equations:

$$\bar{t} = \alpha(t - vx) \qquad\qquad \overline{x} = \alpha(x - vt)$$

Using the invariance of the interval

$$(\Delta s)^2 = (\Delta \overline{s})^2$$

$$-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(\Delta \bar{t})^2 + (\Delta \overline{x})^2 + (\Delta \overline{y})^2 + (\Delta \overline{z})^2$$

If you recall from previously $(\Delta y)^2 = (\Delta \overline{y})^2$ and $(\Delta z)^2 = (\Delta \overline{z})^2$ because the movement of $\overline{\mathscr{O}}$ was in the x direction.

So we get:

$$-(\Delta t)^2 + (\Delta x)^2 = -(\Delta \bar{t})^2 + (\Delta \overline{x})^2$$

Now we plug in $\bar{t} = \alpha(t - vx)$ and $\overline{x} = \alpha(x - vt)$

$$-(\Delta t)^2 + (\Delta x)^2 = -(\alpha(t - vx))^2 + (\alpha(x - vt))^2$$

After simplifying, we get

$$\alpha^2(1 - v^2) = 1$$

$$\alpha = \pm\sqrt{\frac{1}{1 - v^2}}$$

Now we plug α back into the starting equations.

$$\bar{t} = \alpha(t - vx)$$

$$\bar{t} = \frac{t}{\sqrt{1 - v^2}} - \frac{vx}{\sqrt{1 - v^2}}$$

$$\overline{x} = \alpha(x - vt)$$

$$\overline{x} = \frac{x}{\sqrt{1 - v^2}} - \frac{vt}{\sqrt{1 - v^2}}$$

$$\overline{y} = y$$

$$\overline{z} = z$$

This is the Lorentz transformation for a boost of velocity. In this case, it is a boost in the x direction.

1.11 Twin Paradox

The Twin Paradox is a paradox created by special relativity given two twins. There are a pair of twins, Rockette and Earthette, that are 20 years old. One of the twins, Rockette, is sent in a rocket traveling at 99% of the speed of light (2.97 x 10^8 m/s). Both sisters have a clock to measure how long it will be before Rockette returns. Rockette returns to Earth after her clock states that 10 years have passed by. When she arrives, Rockette is only 30 years old. We would presume that Earthette should also be 30 years old. However when you account for the effects of time dilations, Earthette is actually about 91 years old! Would you still consider them to be twins?